RIMS Kôkyûroku Bessatsu
B65 (2017), 109–140

Weighted Morrey spaces–complex interpolation and the boundedness of the Hardy-Littlewood maximal operator

By

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Abstract

The aim of this paper is to address two difficult problems of the Morrey spaces. One is the complex interpolation and another is the behavior of the Hardy-Littlewood maximal operator. Weighted Morrey spaces are difficult to handle due to the following reasons:

1. They are not reflexive.
2. Unlike Lebesgue spaces, there are many non-trivial closed linear subspaces.
3. The norm of the indicator function of the cubes is difficult to calculate.

Nevertheless, it is possible to calculate the second complex interpolation in some special cases. This will allow us to calculate the complex interpolations in such cases. Although we can not always calculate the norm of the indicator function of the cubes, the boundedness of the Hardy-Littlewood maximal operator makes this possible. In this connection, the first half of this artile is devoted to the complex interpolation. In the latter half we investigate what happens if the Hardy-Littlewood maximal operator is bounded on weighted Morrey spaces. As an application, we prove what can we say for the class of weights by using the complex interpolation.

Received September 27, 2016. Revised February 1, 2017.
2010 Mathematics Subject Classification(s): 2010 Mathematics Subject Classification(s):42B35, 42B25,26A33
Key Words: *Complex interpolation, weighted Morrey spaces, Hardy-Littlewood maximal operator.*
The first author is supported financially by Grant-in-Aid for Young Scientists (B) (No. 21740104) Japan Society for the Promotion of Science.
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§ 1. Introduction

Let (X, d, μ) be a metric measure space. In this paper we consider various weighted Morrey spaces. Here and below by a weight w, we mean a measurable function $w : X \to (0, \infty)$ which is positive μ-a.e. and satisfies

$$0 < w(B(x, r)) = \int_{B(x,r)} w(y)\, d\mu(y) < \infty,$$

for all $x \in X$ and $r > 0$. Here $B(x, r)$ stands for the ball centered at x of radius $r > 0$. In particular, we assume $X = \mathrm{supp}(\mu)$ for simplicity. To state our results in full generality, we adopt the following definition of generalized weighted Morrey spaces.

Definition 1.1. Let $q \in [1, \infty)$, $\varphi : X \times (0, \infty) \to (0, \infty)$ be a function and $w, v : X \to [0, \infty)$ be weights. One defines $\mathcal{M}^{\varphi}_q(X, \mu; w, v)$ as the set of all μ-measurable functions f for which the norm

$$\|f\|_{\mathcal{M}^{\varphi}_q(X,\mu;w,v)} := \sup_{x\in X, r>0} \varphi(x, r) \left(\frac{1}{v(B(x, r))} \int_{B(x,r)} |f(y)|^q w(y)\, d\mu(y) \right)^{\frac{1}{q}}$$

is finite.

We chose to work in the framework of this definition of generalized Morrey spaces because it turns out that the underlying geometry is not important for the theory of complex interpolation of Morrey spaces. In particular, the weight v does not affect strongly the results on complex interpolations.

Here are some standard cases we envisage:

Example 1.2. Let $1 \le q \le p < \infty$.

1. The most standard example of (X, d, μ) is the Euclidean space $(\mathbb{R}^n, |\cdot|, dx)$, endowed with the Lebesgue measure, $\varphi(x, r) = |B(x, r)|^{1/p}$, and $v = w = 1$. In this case, we use the symbol $\mathcal{M}^p_q$ to denote $\mathcal{M}^{\varphi}_q(X, \mu; w, v)$, which goes back to the initial work by C. Morrey [24].

2. The generalized Morrey space $\mathcal{M}^{\varphi}_q$ is defined by Nakai [26], where we consider the case (X, d, μ) is the Euclidean space $(\mathbb{R}^n, | \cdot |, dx)$, endowed with the Lebesgue measure, and $v = w = 1$.

3. Since we have freedom in choosing φ, we can consider the parametrized Morrey space $\mathcal{M}^p_q(k, \mu)$ for $k > 0$, the set of all μ-measurable functions f for which the norm

$$\|f\|_{\mathcal{M}^p_q(k,\mu)} := \sup_{x\in X, r>0} \mu(B(x, kr))^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q\, d\mu(y) \right)^{\frac{1}{q}}$$

is finite. By letting $v = w = 1$ and

$$\varphi(x,r) = \mu(B(x,r))^{\frac{1}{q}} \mu(B(x,kr))^{\frac{1}{p}-\frac{1}{q}},$$

we can recover this case. See [32] in the case of the Euclidean space and [7] in the case of the non-doubling metric spaces satisfying the geometrically doubling condition. The Gaussian Morrey space $\mathcal{M}^p_q(\gamma)$ is an example of the spaces considered in [32]. Our results on the complex interpolation will cover the function spaces above. In particular, although we do not work on the quasi-metric spaces, a modification of our result will be available.

4. Let μ be the Lebesgue measure. The Samko type weighted Morrey space $\mathcal{M}^p_q(w) = \mathcal{M}^p_q(w,1)$ [30], whose norm is defined by

$$\|f\|_{\mathcal{M}^p_q(w)} = \|f\|_{\mathcal{M}^p_q(w,1)} := \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q w(y) dy \right)^{\frac{1}{q}}.$$

If one takes $\varphi(x,r) = |B(x,r)|^{\frac{1}{p}}$ and $v = 1$ in Definition 1.1, then one can notice that $\mathcal{M}^p_q(w)$ is an example of $\mathcal{M}^\varphi_q(X,\mu;w,v)$.

5. Let μ be the Lebesgue measure again. The Komori-Shirai type weighted Morrey space $\mathcal{M}^p_q(w) = \mathcal{M}^p_q(w,w)$ [20], whose norm is defined by

$$\|f\|_{\mathcal{M}^p_q(w)} = \|f\|_{\mathcal{M}^p_q(w,w)} := \sup_{x\in\mathbb{R}^n, r>0} w(B(x,r))^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q w(y) dy \right)^{\frac{1}{q}}.$$

If one takes $\varphi(x,r) = w(B(x,r))^{\frac{1}{p}}$ and $v = w$ in Definition 1.1, then one can notice that $\mathcal{M}^p_q(w)$ is an example of $\mathcal{M}^\varphi_q(X,\mu;w,v)$. In particular, many authors investigated the case when $\varphi(x,r)v(B(x,r))^{-\frac{1}{p}} = w(B(x,r))^{\frac{\kappa}{p}}$. See [10, 17, 18, 37, 38, 39, 40, 42, 45].

6. Let μ be the Lebesgue measure again. Then we can slightly generalize the above definition to have $\mathcal{M}^p_q(w_1,w_2)$ [20], whose norm is defined by

$$\|f\|_{\mathcal{M}^p_q(w_1,w_2)} := \sup_{x\in\mathbb{R}^n, r>0} w_2(B(x,r))^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q w_2(y) dy \right)^{\frac{1}{q}}.$$

7. As a special case of the weights, we can consider the power weight $w(x) = |x|^\alpha$.

8. As an example of the function φ we can list the weight of the type

$$\varphi(x) = \prod_{k=1}^n \omega_k(|x - x_k|),$$

$x_1, x_2, \ldots, x_n$ is a fixed point and $\omega_1, \omega_2, \ldots, \omega_n$ are suitable functions. See [19] for example.

9. One can also consider the mixture of in the above to consider generalized weighted Morrey spaces.

Although some important properties of Morrey spaces became clear recently, it is still difficult to investigate Morrey spaces. Let us review some recent progress on Morrey spaces to see why the interpolation of Morrey spaces are difficult. Let $1 < q < p < \infty$.

1. The Morrey space $\mathcal{M}^p_q$ is not reflexive; see [34, Example 5.2] and [41, Theorem 1.3].
2. The Morrey space $\mathcal{M}^p_q$ does not have $C^\infty_{\rm c}$ as a dense closed subspace; see [36, Proposition 2.16].
3. The Morrey space $\mathcal{M}^p_q$ is not separable; see [36, Proposition 2.16].
4. The Morrey space $\mathcal{M}^p_q$ is not included in $L^1 + L^\infty$; see [16, Section 6].

Not only the complex interpolation but also the real interpolation is difficult. However, Burenkov and Nursultanov obtained the description of the interpolation of local Morrey spaces [5]. Note that local Morrey spaces are the modification of Morrey spaces. We also refer to [27] for the extension of the results in [5] to B_σ spaces. We do not go into the detail of the interpolations of Morrey spaces here. In Section 2.3, we recall the progress of the complex interpolation of Morrey spaces.

Taking the supremum over all cubes seems to make things more difficult. Due to this fact, it is difficult to calculate or estimate the norm of the indicator function of the cubes, for example. Since we can not estimate of the norm of such functions, it is difficult to estimate the norm of any other function. Thus, it seems difficult to describe the necessary and sufficient conditions for the Hardy-Littlewood maximal operator M, which is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y) \quad (x \in X)$$

for a μ-measurable function f, to be bounded on $\mathcal{M}^\varphi_q(X,\mu;w,v)$.

In view of the beautiful theory of A_p-weights for Lebesgue spaces [25], it seems natural to propose the following problem:

Problem 1.3. *Look for the condition for which there exists a constant $C > 0$ such that $\|Mf\|_{\mathcal{M}^\varphi_q(X,\mu;w,v)} \le C\|f\|_{\mathcal{M}^\varphi_q(X,\mu;w,v)}$ holds for all $f \in \mathcal{M}^\varphi_q(X,\mu;w,v)$.*

One trivial necessary condition is that

$$\frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y) \times \|\chi_Q\|_{\mathcal{M}^\varphi_q(X,\mu;w,v)} \le C\|f\|_{\mathcal{M}^\varphi_q(X,\mu;w,v)} \tag{1.1}$$

for all $f \in \mathcal{M}^{\varphi}_q(X,\mu;w,v)$. The condition (1.1) seems attractive because this condition is equivalent to the A_p-condition in the case of $(X,d,\mu) = (\mathbb{R}^n, |\cdot|, dx)$ and $1 < p = q < \infty$ [9, Chapter 7]. So, we conjecture the following:

Conjecture 1.4. *The condition* (1.1) *is sufficient for the Hardy-Littlewood maximal operator to be bounded in the case of* $(X,d,\mu) = (\mathbb{R}^n, |\cdot|, dx)$ *and* $1 < q < p < \infty$.

Although Problem 1.3 is still open, we can say something more about this problem and Conjecture 1.4.

We organize the remaining part of this paper as follows: In Section 2, we review the definition of the complex interpolation functors and then we formulate the main results on the complex interpolation. Our strategy to calculate the complex interpolation spaces is to consider the second complex interpolation of the spaces first and then move on to the first complex interpolation. Section 3 contains the proof of the results in Section 2. Section 4 considers Problem 1.3. We describe what is known about this problem and then we apply the result in Section 2 to have a related result.

§ 2. Complex interpolation of Morrey spaces

§ 2.1. Two interpolation functors

Let $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\bar{S} := \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$. We adopt the following definition of two complex interpolation functors:

Definition 2.1 (Calderón's first complex interpolation space)**.** Let (X_0, X_1) be a compatible couple of Banach spaces.

1. Define $\mathcal{F}(X_0,X_1)$ as the set of all functions $F : \overline{S} \to X_0 + X_1$ such that
   (a) F is continuous on $\overline{S}$ and $\sup\limits_{z \in \overline{S}} \|F(z)\|_{X_0+X_1} < \infty$,
   (b) F is holomorphic on S,
   (c) the functions $t \in \mathbb{R} \mapsto F(j+it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0,1$.

   The space $\mathcal{F}(X_0,X_1)$ is equipped with the norm
   $$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1}\right\}.$$
2. Let $\theta \in (0,1)$. Define the complex interpolation space $[X_0,X_1]_\theta$ with respect to (X_0,X_1) to be the set of all functions $x \in X_0 + X_1$ such that $x = F(\theta)$ for some $F \in \mathcal{F}(X_0,X_1)$. The norm on $[X_0,X_1]_\theta$ is defined by
   $$\|x\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : x = F(\theta) \text{ for some } F \in \mathcal{F}(X_0,X_1)\}.$$

Let Y be a Banach space. We let $\mathrm{Lip}(\mathbb{R}, Y)$ be the set of all continuous functions $f : \mathbb{R} \to Y$ for which the quantity $\|f\|_{\mathrm{Lip}(\mathbb{R},Y)} := \sup_{-\infty<s<t<\infty} \dfrac{\|f(t) - f(s)\|_Y}{|t - s|}$ is finite.

Definition 2.2 (Calderón's second complex interpolation space)**.** Suppose that $\overline{X} = (X_0, X_1)$ is a compatible couple of Banach spaces.

1. Define $\mathcal{G}(X_0, X_1)$ as the set of all functions $F : \overline{S} \to X_0 + X_1$ such that
   (a) F is continuous on $\overline{S}$ and $\sup_{z\in\overline{S}} \left\| \frac{F(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$,
   (b) F is holomorphic on S,
   (c) the functions $t \in \mathbb{R} \mapsto F(j + it) - F(j) \in X_j$ are Lipschitz continuous on $\mathbb{R}$ for $j = 0, 1$.

   The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm

   $$\|F\|_{\mathcal{G}(X_0,X_1)} := \max\left\{\|F(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \|F(1 + i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\right\} \tag{2.1}$$

2. Let $\theta \in (0, 1)$. Define the complex interpolation space $[X_0, X_1]^\theta$ with respect to (X_0, X_1) to be the set of all functions $x \in X_0 + X_1$ such that $x = F'(\theta)$ for some $F \in \mathcal{G}(X_0, X_1)$. The norm on $[X_0, X_1]^\theta$ is defined by

   $$\|x\|_{[X_0,X_1]^\theta} := \inf\{\|F\|_{\mathcal{G}(X_0,X_1)} : x = F'(\theta) \text{ for some } F \in \mathcal{G}(X_0, X_1)\}.$$

To describe our main results in this paper, we write

$$E_{R,0} := \left\{x \in X \,:\, |f(x)|^{q_0-q}\frac{w_0(x)}{w(x)} \geq R\right\} = \left\{x \in X \,:\, |f(x)|^{q_0-q_1}\frac{w_0(x)}{w_1(x)} \geq R^{\frac{q_1}{q\theta}}\right\} \tag{2.2}$$

$$E_{R,1} := \left\{x \in X \,:\, |f(x)|^{q_1-q}\frac{w_1(x)}{w(x)} \geq R\right\} = \left\{x \in X \,:\, |f(x)|^{q_1-q_0}\frac{w_1(x)}{w_0(x)} \geq R^{\frac{q_0}{q(1-\theta)}}\right\} \tag{2.3}$$

$$E_R := E_{R,0} \cup E_{R,1}, \tag{2.4}$$

when we have measurable functions f, w_0, w_1, and w satisfying

$$w(x) := w_0(x)^{\frac{(1-\theta)q}{q_0}} w_1(x)^{\frac{\theta q}{q_1}} \quad (x \in X). \tag{2.5}$$

The set E_R will play the role of the level set of f in the weighted setting. Note that this does not depend on v. Define

$$f_R := f(1 - \chi_{E_R}) \tag{2.6}$$

for $R > 0$ and $f \in \mathcal{M}^{\varphi}_q(X, \mu; w, v)$. We are interested in the condition:

$$f = \lim_{R\to\infty} f_R \text{ in } \mathcal{M}^{\varphi}_q(X, \mu; w, v). \tag{2.7}$$

In fact, it will turn out that (2.7) will be the standard approximation of the weighted Morrey spaces. Our first main result is as follows:

Theorem 2.3. *Let $0 < \theta < 1$ and $q_0, q_1 \in [1, \infty)$. Let $\varphi_0, \varphi_1 : X \times (0, \infty) \to (0, \infty)$ be functions and $w_0, w_1, v : X \to [0, \infty)$ be weights satisfying* (2.5)*. Assume that q_0, q_1, φ_0 and φ_1 satisfy*

$$\varphi_0(x, t)^{q_0} = \varphi_1(x, t)^{q_1} \quad (x \in X, t > 0). \tag{2.8}$$

Define q and φ by

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \text{ and } \varphi(x, t) := \varphi_0(x, t)^{1-\theta}\varphi_1(x, t)^{\theta}. \tag{2.9}$$

Then

$$[\mathcal{M}^{\varphi_0}_{q_0}(X, \mu; w_0, v), \mathcal{M}^{\varphi_1}_{q_1}(X, \mu; w_1, v)]_\theta = \{f \in \mathcal{M}^{\varphi}_q(X, \mu; w, v) : \text{ (2.7) } \textit{holds}\}, \tag{2.10}$$

$$[\mathcal{M}^{\varphi_0}_{q_0}(X, \mu; w_0, v), \mathcal{M}^{\varphi_1}_{q_1}(X, \mu; w_1, v)]^\theta = \mathcal{M}^{\varphi}_q(X, \mu; w, v). \tag{2.11}$$

To investigate the role of these two functors, we further consider the following closed subspaces;

Definition 2.4. Let $q \in [1, \infty)$, $\varphi : X \times (0, \infty) \to (0, \infty)$ be a function, and $w, v, \tilde{w} : X \to [0, \infty)$ be weights. Write $U_{\tilde{w}} := \{f \in L^0(\mu) : f\tilde{w} \in L^\infty(\mu)\}$.

1. Denote by $L^0(\mu)$ the set of all μ-measurable functions.
2. Denote by $L^0_{\mathrm{c}}(\mu)$ the set of all μ-measurable functions having bounded support.
3. Let $U \subset L^0(\mu)$ be a linear subspace with the lattice property: $|g| \le |f|$ and $f \in U$ implies $g \in U$. One defines the closed subspace $U\mathcal{M}^{\varphi}_q(X, \mu; w, v)$, called *closed subspace of $\mathcal{M}^{\varphi}_q(X, \mu; w, v)$ generated by U,* as the closure of $U \cap \mathcal{M}^{\varphi}_q(X, \mu; w, v)$ in $\mathcal{M}^{\varphi}_q(X, \mu; w, v)$.
4. The *bar subspace $\overline{\mathcal{M}}^{\varphi}_q(X, \mu; w, v)$ of $\mathcal{M}^{\varphi}_q(X, \mu; w, v)$* is defined to be $U\mathcal{M}^{\varphi}_q(X, \mu; w, v)$ with $U = L^\infty(\mu)$.
5. The *star subspace* $\overset{*}{\mathcal{M}}{}^{\varphi}_q(X, \mu; w, v)$ of $\mathcal{M}^{\varphi}_q(X, \mu; w, v)$ is $U\mathcal{M}^{\varphi}_q(X, \mu; w, v)$, where $U = L^0_{\mathrm{c}}(\mu)$.
6. The *tilde subspace* $\widetilde{\mathcal{M}}{}^{\varphi}_q(X, \mu; w, v)$ *of* $\mathcal{M}^{\varphi}_q(X, \mu; w, v)$ is $U\mathcal{M}^{\varphi}_q(X, \mu; w, v)$, where $U = L^\infty_{\mathrm{c}}(\mu)$.

7. The *bar subspace* $\overline{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v;\ \mathrm{rel}\ \tilde{w})$ *of* $\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$ *relative to* $\tilde{w}$ is defined to be the closure of the set $U_{\tilde{w}}$ in $\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$.

8. The *tilde subspace* $\widetilde{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v;\ \mathrm{rel}\ \tilde{w})$ *of* $\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$ *relative to* $\tilde{w}$ is defined to be the closure of the set $U_{\tilde{w}} \cap L^0_{\mathrm{c}}(\mu)$ in $\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$.

We now describe the second complex interpolation of these closed subspaces:

Theorem 2.5. *Maintain the same assumption on* $q_0, q_1, q, \varphi_0, \varphi_1, \varphi, w_0, w_1$, *and* w *as Theorem 2.3. Define*

$$A := |f|^{\frac{q}{q_1}-\frac{q}{q_0}} w^{\frac{1}{q_1}-\frac{1}{q_0}} w_0^{\frac{1}{q_0}} w_1^{-\frac{1}{q_1}} \left(= |f|^{q_0-q_1}\frac{w_0}{w_1}\right)^{\frac{q}{q_0q_1}}, \quad \tilde{w} = \left(\frac{w_0}{w_1}\right)^{\frac{1}{q_0-q_1}}. \tag{2.12}$$

1. *For the interpolation of the bar subspaces relative to weights, we have*

$$\begin{aligned}(2.13)\quad & [\overline{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \overline{\mathcal{M}}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})]^{\theta} \\ & = [\overline{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta} = \mathcal{M}^{\varphi}_{q}(X,\mu;w,v).\end{aligned}$$

In particular, when $w_0 = w_1 = w$, *we have*

$$\begin{aligned}(2.14)\quad & [\overline{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overline{\mathcal{M}}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta} \\ & = [\overline{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta} = \mathcal{M}^{\varphi}_{q}(X,\mu;w,v).\end{aligned}$$

2. *Let* $G^{(j)} = G^{(j)}_{f,A,w_0,w_1,w} := \chi_{\{\frac{1}{R}\le A\le R\}}|f|^{\frac{q}{q_j}} w^{\frac{1}{q_j}} w_j^{-\frac{1}{q_j}}$ *for* $f \in \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$. *For the interpolation of the tilde subspaces relative to weights, we have*

$$\begin{aligned}(2.15)\quad & \bigcap_{\substack{R>1\\ j\in\{0,1\}}} \left\{ f \in \mathcal{M}^{\varphi}_{q}(X,\mu;w,v) : G^{(j)} \in \overset{*}{\mathcal{M}}{}^{\varphi_j}_{q_j}(X,\mu;w_j,v)\right\} \\ & \subseteq [\widetilde{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \widetilde{\mathcal{M}}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})]^{\theta} \\ & \subseteq [\widetilde{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta} \\ & \subseteq [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta} \\ & \subseteq \mathcal{M}^{\varphi}_{q}(X,\mu;w,v) \cap \overline{\overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v)}^{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)+\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}.\end{aligned}$$

In particular, when $w_0 = w_1 = w$, we have

$$\begin{aligned}
(2.16)\qquad &\bigcap_{R>1}\left\{f \in \mathcal{M}^{\varphi}_{q}(X,\mu;w,v) : \chi_{[R^{-1},R]}(|f|) \in \overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v)\right\}\\
&= [\widetilde{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}\\
&= [\widetilde{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \widetilde{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}\\
&= [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}\\
&= [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}\\
&= \overline{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v) \cap \overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v)}^{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)+\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}.
\end{aligned}$$

We move on to the first complex interpolation of the closed subspaces of Morrey spaces.

Theorem 2.6. *Maintain the same assumption on $q_0, q_1, q, \varphi_0, \varphi_1, \varphi, w_0, w_1$ and w as Theorem 2.3. Let $\tilde{w}$ be a weight defined by* (2.12)*.*

1. *The description of the star subspaces is as follows*:

$$\begin{aligned}
(2.17)\qquad &[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_{\theta}\\
&= [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_{\theta}\\
&= \{f \in \overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v) \ :\ (2.7)\ \textit{holds}\}.
\end{aligned}$$

2. *The description of the bar subspaces relative to weights is as follows*:

$$\begin{aligned}
(2.18)\qquad &[\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \overline{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})]_{\theta}\\
&= [\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_{\theta}\\
&= \{f \in \overline{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v;\ \mathrm{rel}\ \tilde{w}) \ :\ (2.7)\ \textit{holds}\}.
\end{aligned}$$

In particular, when $w_0 = w_1 = w$, we have the following description of the bar subspaces:

$$\begin{aligned}
(2.19)\qquad &[\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overline{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_{\theta}\\
&= [\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_{\theta}\\
&= \{f \in \overline{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v) \ :\ (2.7)\ \textit{holds}\}.
\end{aligned}$$

3. *The description of the tilde subspaces relative to weights is as follows*:

$$\begin{aligned}
(2.20)\qquad &[\widetilde{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \widetilde{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})]_{\theta}\\
&= [\widetilde{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_{\theta}\\
&= \{f \in \widetilde{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v;\ \mathrm{rel}\ \tilde{w}) \ :\ (2.7)\ \textit{holds}\}.
\end{aligned}$$

In particular, when $w_0 = w_1 = w$, we have the following description of the tilde subspaces:

$$\begin{aligned}(2.21)\qquad &[\widetilde{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\widetilde{\mathcal{M}}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta\\ &=[\widetilde{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta\\ &=\{f\in\widetilde{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v)\ :\ (2.7)\ \textit{holds}\}.\end{aligned}$$

§ 2.2. Auxiliary lemmas

As for the five closed subspaces, we have the following characterization:

Lemma 2.7. *Let $\tilde{w}$ be a weight. For $f \in \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$, we write*

$$F_R(\tilde{w})=\{x\in X\ :\ |f(x)|\tilde{w}(x)\le R\},\quad F_R=F_R(1).$$

Fix a point $\mathrm{o}\in X$*, which is called the base point. Write* $B(R)=\{y\in X\ :\ d(\mathrm{o},y)\le R\}$*. Then*

$$\begin{aligned}(2.22)\quad &\overline{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v)\\ &=\left\{f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\ :\ \lim_{R\to\infty}f\chi_{F_R}=f\ \text{in}\ \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\right\}.\\ (2.23)\quad &\overline{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v;\ \mathrm{rel}\ \tilde{w})\\ &=\left\{f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\ :\ \lim_{R\to\infty}f\chi_{F_R(\tilde{w})}=f\ \text{in}\ \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\right\}.\\ (2.24)\quad &\overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v)\\ &=\left\{f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\ :\ \lim_{R\to\infty}f\chi_{B(R)}=f\ \text{in}\ \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\right\}.\\ (2.25)\quad &\widetilde{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v;\ \mathrm{rel}\ \tilde{w})\\ &=\left\{f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\ :\ \lim_{R\to\infty}f\chi_{B(R)\cap F_R(\tilde{w})}=f\ \text{in}\ \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\right\}.\\ (2.26)\quad &\widetilde{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v)\\ &=\left\{f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\ :\ \lim_{R\to\infty}f\chi_{B(R)\cap F_R}=f\ \text{in}\ \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\right\}.\end{aligned}$$

Proof. In (2.22)–(2.26), "$\supset$" is easy to prove. To prove "$\subset$", we mimic the proof of our earlier results. See [16, Lemma 2.6] for $\overline{\mathcal{M}}^{\varphi}_{q}(X,\mu;w,v)$ and [14, Theorem 1.3] for (2.26). Let us prove (2.23). Let $f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v;\mathrm{rel}\tilde{w})$. For every $\varepsilon>0$, choose $g=g_\varepsilon\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$ such that $g\tilde{w}\in L^\infty(X,\mu)$ and that $\|f-g\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}<\frac{\varepsilon}{2}$.

Set $C := \|g\tilde{w}\|_{L^\infty(X,\mu)}$. Observe that, for each $R > 2C$, we have

$$\begin{aligned}|f - f\chi_{F_R(\tilde{w})}| &\le |f-g| + |g(1-\chi_{F_R(\tilde{w})})|\\ &\le |f-g| + \frac{R}{2\tilde{w}}(1-\chi_{F_R(\tilde{w})})\\ &\le |f-g| + \frac{1}{2}|f - f\chi_{F_R(\tilde{w})}|.\end{aligned}$$

Consequently, for every $R > 2C$, $\|f - f\chi_{F_R(\tilde{w})}\|_{\mathcal{M}_q^\varphi(X,\mu;w,v)} \le 2\|f-g\|_{\mathcal{M}_q^\varphi(X,\mu;w,v)} < \varepsilon$. Thus, we have showed that (2.23) holds.

For (2.24), we adapt the proof of [14, Theorem 1.3]. Let $f \in \overset{*}{\mathcal{M}}{}_q^\varphi(X,\mu;w,v)$. Given $\varepsilon > 0$, there exists $g_\varepsilon \in L^0_{\rm c} \cap \mathcal{M}_q^\varphi(X,\mu;w,v)$ such that

$$\|f - g_\varepsilon\|_{\mathcal{M}_q^\varphi(X,\mu;w,v)} < \varepsilon. \tag{2.27}$$

For any $R > 0$, we have $|f - f\chi_{B(R)}| \le |g_\varepsilon(1-\chi_{B(R)})| + |f - g_\varepsilon|$. Choose $R_\varepsilon > 0$ such that $\operatorname{supp}(g_\varepsilon) \subset B(R_\varepsilon)$. Then, $|f - f\chi_{B(R)}| \le |f - g_\varepsilon|$ for all $R > R_\varepsilon$, and hence

$$\|f - f\chi_{B(R)}\|_{\mathcal{M}_q^\varphi(X,\mu;w,v)} \le \|f - g_\varepsilon\|_{\mathcal{M}_q^\varphi(X,\mu;w,v)} < \varepsilon.$$

Thus, we have proved (2.24). We can prove (2.25) combining the argument in the proof of (2.23) and (2.24). □

We invoke the Hölder inequality for generalized weighted Morrey spaces as follows:

Lemma 2.8. *Keep using the same assumption on* $q_0, q_1, q, \varphi_0, \varphi_1, \varphi, w_0, w_1$, *and* w *as Theorem 2.3. If* $f \in \mathcal{M}_{q_0}^{\varphi_0}(X,\mu;w_0,v) \cap \mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v)$, *then*

$$\|f\|_{\mathcal{M}_q^\varphi(X,\mu;w,v)} \le \|f\|^{1-\theta}_{\mathcal{M}_{q_0}^{\varphi_0}(X,\mu;w_0,v)} \|f\|^{\theta}_{\mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v)}. \tag{2.28}$$

Proof. The proof uses (2.9) and the Hölder inequality. □

Combining inequality (2.28) and Lemma 2.7, we obtain the following inclusions:

Lemma 2.9. *Keep using the same assumption on* $q_0, q_1, q, \varphi_0, \varphi_1, \varphi, w_0, w_1$, *and* w *as Theorem 2.3. Let also* $\tilde{w}$ *be a weight. Then*

$$\begin{gathered}\overline{\mathcal{M}}{}_{q_0}^{\varphi_0}(X,\mu;w_0,v;\ \text{rel } \tilde{w}) \cap \mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v) \subseteq \overline{\mathcal{M}}{}_q^{\varphi}(X,\mu;w,v;\ \text{rel } \tilde{w});\\ \overset{*}{\mathcal{M}}{}_{q_0}^{\varphi_0}(X,\mu;w_0,v) \cap \mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v) \subseteq \overset{*}{\mathcal{M}}{}_q^{\varphi}(X,\mu;w,v);\\ \widetilde{\mathcal{M}}{}_{q_0}^{\varphi_0}(X,\mu;w_0,v;\ \text{rel } \tilde{w}) \cap \mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v) \subseteq \widetilde{\mathcal{M}}{}_q^{\varphi}(X,\mu;w,v;\ \text{rel } \tilde{w}).\end{gathered}$$

Now, for given $f \in \mathcal{M}^{\varphi}_q(X,\mu;w,v)$, we construct the second complex interpolation functor as follows: Define

$$F(z) := \operatorname{sgn}(f)|f|^{q\frac{1-z}{q_0}+q\frac{z}{q_1}} w^{\frac{1-z}{q_0}+\frac{z}{q_1}} w_0^{-\frac{1-z}{q_0}} w_1^{-\frac{z}{q_1}}, \quad G(z) := \int_{\theta\to z} F(h)\,dh, \tag{2.29}$$

where $\theta \to z$ stands for any C^1-curve in $\overline{S}$ from θ to z. We set

$$A := |f|^{\frac{q}{q_1}-\frac{q}{q_0}} w^{\frac{1}{q_1}-\frac{1}{q_0}} w_0^{\frac{1}{q_0}} w_1^{-\frac{1}{q_1}} = \left(|f|^{q_0-q_1}\frac{w_0}{w_1}\right)^{\frac{q}{q_0q_1}} \tag{2.30}$$

and

$$F_0 := \chi_{\{A\le 1\}}F,\ F_1 := F - F_0,\ G_0 := \chi_{\{A\le 1\}}G,\ G_1 := G - G_0. \tag{2.31}$$

We prove several lemmas as follows:

Lemma 2.10. *For all $z \in \overline{S}$, $G(z) \in \mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$. Moreover,*

$$\sup_{z\in\overline{S}} \left\|\frac{G(z)}{1+|z|}\right\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)+\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)} < \infty. \tag{2.32}$$

Proof. Let $z \in \overline{S}$. Since $\mathrm{Re}(z) \ge 0$, we have

$$|F_0(z)| = \chi_{\{A\le1\}}|f|^{\frac{q}{q_0}}\left(\frac{w}{w_0}\right)^{\frac{1}{q_0}} A^{\mathrm{Re}(z)} \le |f|^{\frac{q}{q_0}}\left(\frac{w}{w_0}\right)^{\frac{1}{q_0}}. \tag{2.33}$$

Therefore,

$$|G_0(z)| \le |z-\theta||f|^{\frac{q}{q_0}}\left(\frac{w}{w_0}\right)^{\frac{1}{q_0}} \le (1+|z|)|f|^{\frac{q}{q_0}}\left(\frac{w}{w_0}\right)^{\frac{1}{q_0}},$$

which yields $\|G_0(z)\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)} \le (1+|z|)\|f\|^{\frac{q}{q_0}}_{\mathcal{M}^{\varphi}_q(X,\mu;w,v)}$. Similarly, we can prove $\|G_1(z)\|_{\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)} \le (1+|z|)\|f\|^{\frac{q}{q_1}}_{\mathcal{M}^{\varphi}_q(X,\mu;w,v)}$. Combining the norm estimates for $G_0(z)$ and $G_1(z)$, we get

$$\|G(z)\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)+\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)} \le (1+|z|)\left(\|f\|^{\frac{q}{q_0}}_{\mathcal{M}^{\varphi}_q(X,\mu;w,v)} + \|f\|^{\frac{q}{q_1}}_{\mathcal{M}^{\varphi}_q(X,\mu;w,v)}\right).$$

and hence, (2.32) holds. □

Lemma 2.11. *Let $q_0 > q_1$. Then $G : \overline{S} \to \mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$ is continuous. More precisely, $G_j : \overline{S} \to \mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v)$ is continuous.*

Proof. Let us concentrate on G_0; the proof for G_1 is similar. Fix $z, z_0 \in \bar{S}$. By using (2.33), we obtain $|G_0(z) - G_0(z_0)| \le |f|^{\frac{q}{q_0}} \left(\frac{w}{w_0}\right)^{\frac{1}{q_0}} |z - z_0|$. Consequently,

$$\|G_0(z) - G_0(z_0)\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)} \le |z - z_0| \|f\|^{\frac{q}{q_0}}_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}.$$

Thus, $\|G_0(z) - G_0(z_0)\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)} = O(|z - z_0|)$ as $z \to z_0$, as was to be shown. □

Lemma 2.12. *The function $G|S : S \to \mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$ is holomorphic and $G'(z) = F(z)$ in $\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$ for all $z \in S$. In particular,*

$$f = G'(\theta) = \lim_{h\to 0} \frac{G(h+\theta) - G(\theta)}{h} \text{ in } \mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v). \tag{2.34}$$

Proof. By virtue of (2.33) and its analog for $F_1(z)$, for every $z \in S$, we have

$$F(z) \in \mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v).$$

Let $0 < \varepsilon \ll 1$ and $S_\varepsilon := \{z \in S : \varepsilon < \mathrm{Re} z < 1 - \varepsilon\}$. We fix $z \in S_\varepsilon$. Suppose $h \in \mathbb{C}$ satisfies $|h| < \frac{\varepsilon}{2}$ and $z + h \in S$. Consider the functions defined by (2.30). Since $\mathrm{Re} z > \varepsilon$ and $|h| < \frac{\varepsilon}{2}$, we have

$$\begin{aligned}
\left|\frac{G_0(z+h) - G_0(z)}{h} - F_0(z)\right| &= \chi_{\{A\le 1\}}|F(z)| \left|\frac{A^h - 1}{h \log A} - 1\right| \\
&\le \chi_{\{A\le 1\}}|f|^{\frac{q}{q_0}} \left(\frac{w_0}{w}\right)^{\frac{1}{q_0}} A^{\mathrm{Re} z}|h \log A| e^{|h \log A|} \\
&\le \chi_{\{A\le 1\}}|f|^{\frac{q}{q_0}} \left(\frac{w_0}{w}\right)^{\frac{1}{q_0}} A^{\varepsilon}|h \log A| A^{-\frac{\varepsilon}{2}} \\
&\le \frac{2|h|}{\varepsilon e}|f|^{\frac{q}{q_0}} \left(\frac{w_0}{w}\right)^{\frac{1}{q_0}}.
\end{aligned}$$

Therefore,

$$\left\|\frac{G_0(z+h) - G_0(z)}{h} - F_0(z)\right\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)} \le \frac{2|h|}{\varepsilon e} \|f\|^{\frac{q}{q_0}}_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}. \tag{2.35}$$

Hence, $G_0'(z) = F_0(z)$ in $\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)$. Likewise, $G_1'(z) = F_1(z)$ in $\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w,v)$. As a result, $G'(z) = F(z)$ in $\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w,v)$. Since ε is arbitrary, we conclude that G is holomorphic in S and $G'(z) = F(z)$ for every $z \in S$. In particular, evaluating this relation at $z = \theta$, we obtain (2.34). □

Lemma 2.13. *For all $j = 0, 1$ and $t, t' \in \mathbb{R}$,*

$$\|G(j + it') - G(j + it)\|_{\mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v)} \le |t - t'| (\|f\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)})^{\frac{q}{q_j}},$$

Proof. As before, by using the triangle inequality for the complex line integral and

$$|F(j+iu)| \le |f|^{\frac{q}{q_j}} \left(\frac{w}{w_j}\right)^{\frac{1}{q_j}},$$

for every $u \in \mathbb{R}$, we obtain $|G(j+it') - G(j+it)|^{q_j} w_j \le |f|^q w |t'-t|^q$, so the result is immediate. □

Concerning the following construction, we have the following helpful remark:

Remark. Let $f \in \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$ and $R > 0$. Define $G_R := \chi_{[R^{-1},R]}(|A|)G$, $G_{0,R} := G_R = \chi_{[R^{-1},R]}(|A|)G_0$ and $G_{1,R} := G_R = \chi_{[R^{-1},R]}(|A|)G_1$.

(*A*) As for the function G, if we truncate it at the level set $\{R^{-1} \le |A| \le R\}$, we have

$$\|G_j - G_{j,R}\|_{\mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v)} \le \frac{(\|f\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)})^{\frac{q_0}{q_j}}}{\log R}$$

and

$$\begin{aligned}&\|G_j(j+it') - G_{j,R}(j+it') - G_j(j+it) + G_{j,R}(j+it)\|_{\mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w,v)}\\ &\le |t-t'| \frac{(\|f\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)})^{\frac{q}{q_j}}}{\log R}.\end{aligned}$$

Thus, $G_R \to G$ as $R \to \infty$ in $\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v))$.

(*B*) Define $\tilde{w}$ by (2.12). Similar to above, one can check $G_0 \in \overline{\mathcal{M}}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w})$ and $G_1 \in \overline{\mathcal{M}}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})$.

For the complex interpolation of closed subspaces, we prove the following lemmas:

Lemma 2.14. *Keep using the same assumption as in Theorem 2.3. Then*

$$\begin{aligned}&\overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v) \cap \overline{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) \cap \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}\\ &\subseteq \overline{\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v) \cap \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}}.\end{aligned} \tag{2.36}$$

Proof. Let

$$f \in \overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v) \cap \overline{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) \cap \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}.$$

Take $\{f_k\}_{k=1}^{\infty} \subseteq \mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) \cap \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$ convergent to f in $\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$. Define $f_{k,R} := \chi_{B(R)} f_k$ for $R \in \mathbb{N}$. Note that, the following inequality

$$\|f - f_{k,R}\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)} \le \|f - \chi_{B(R)} f\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)} + \|f - f_k\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)},$$

implies

$$\lim_{k,R\to\infty} \|f - f_{k,R}\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)} = 0. \tag{2.37}$$

For each $z \in \overline{S}$, define

$$F_{k,R}(z) := \operatorname{sgn}(f_{k,R})|f_{k,R}|^{q\frac{1-z}{q_0}+q\frac{z}{q_1}} \frac{w^{\frac{1-z}{q_0}+\frac{z}{q_1}}}{w_0^{\frac{1-z}{q_0}} w_1^{\frac{z}{q_1}}} \text{ and } G_{k,R}(z) := \int_{\theta}^{z} F_{k,R}(u)\, du. \tag{2.38}$$

By virtue of Lemmas 2.10-2.13, we have $G_{k,R} \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v))$. Since $\operatorname{supp}(f_{k,R}) \subseteq B(R)$,

$$|G_{k,R}(z)| \le (1+|z|) \left(|f_{k,R}|^{\frac{q}{q_0}} \left(\frac{w}{w_0}\right)^{\frac{1}{q_0}} + |f_{k,R}|^{\frac{q}{q_1}} \left(\frac{w}{w_1}\right)^{\frac{1}{q_1}} \right),$$

and

$$|G_{k,R}(j+it) - G_{k,R}(j)| \le |t| \cdot |f_{k,R}|^{\frac{q}{q_j}} \left(\frac{w}{w_j}\right)^{\frac{1}{q_j}}$$

for every $j \in \{0,1\}$ and $t \in \mathbb{R}$, we have $G_{k,R}(z) \in \overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$ and $G_{k,R}(j+it) - G_{k,R}(j) \in \overset{*}{\mathcal{M}}{}^{\varphi_j}_{q_j}(X,\mu;w_j,v)$. Therefore,

$$G_{k,R} \in \mathcal{G}(\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)).$$

Since $f_{k,R} = G'_{k,R}(\theta)$, we have

$$\begin{aligned} \|f_{k,R}\|_{[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}} &\le \|G_{k,R}\|_{\mathcal{G}(\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v))} \\ &\le \max_{j=0,1} \|f_{k,R}\|^{\frac{q}{q_j}}_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}. \end{aligned} \tag{2.39}$$

Since $\{f_{k,R}\}_{k,R\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$, $\{f_{k,R}\}_{k,R\in\mathbb{N}}$ is a Cauchy sequence in $[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}$. Hence,

$$\lim_{k,R\to\infty} \|f_{k,R} - g\|_{[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}} = 0 \tag{2.40}$$

for some $g \in [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}$.

Combining (2.37), (2.40), and

$$[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta} \subseteq \mathcal{M}^{\varphi}_{q}(X,\mu;w,v),$$

we conclude that $f = g$, so

$$f \in \overline{\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v) \cap \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^{\theta}},$$

as desired. $\square$

Lemma 2.15. *Keep using the same assumption as in Theorem 2.3. Let $f \in \mathcal{M}^{\varphi}_p(X,\mu;w,v)$ and define $\tilde{w}$ by (2.12). Then*

$$\chi_{\{R^{-1}\le A\le R\}}|f|^{\frac{q}{q_j}}\left(\frac{w}{w_j}\right)^{\frac{1}{q_j}} \in \mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v;\ \mathrm{rel}\ \tilde{w}) \quad (j=0,1). \tag{2.41}$$

Proof. Without loss of generality, assume that $q_0 > q_1$. An arithmetic shows that

$$\left(\frac{w}{w_j}\right)^{\frac{q}{q_j}}\left(\frac{w_0}{w_1}\right)^{\frac{1}{q_0-q_1}} = \left(\frac{w_0}{w_1}\right)^{\frac{q}{q_j(q_0-q_1)}}$$

Therefore, in view of (2.30), on $\{R^{-1}\le A\le R\}$, we have

$$|f|^{\frac{q}{q_j}}\left(\frac{w}{w_j}\right)^{\frac{1}{q_j}}\left(\frac{w_0}{w_1}\right)^{\frac{1}{q_0-q_1}} = A^{\frac{q_{1-j}}{q_0-q_1}} \le R^{\frac{q_{1-j}}{q_0-q_1}}.$$

Thus, (2.41) follows. □

Next, we collect some information on the Poisson integral. We define

$$\mu_0(t) := \frac{\sin(\pi\theta)}{2[\cosh(\pi t)-\cos(\pi\theta)]} \text{ and } \mu_1(t) := \frac{\sin(\pi\theta)}{2[\cosh(\pi t)+\cos(\pi\theta)]}. \tag{2.42}$$

Note that $\|\mu_0\|_{L^1(\mathbb{R})} = 1-\theta$ and $\|\mu_1\|_{L^1(\mathbb{R})} = \theta$. We need two other lemmas on complex analysis.

Lemma 2.16. [11, Lemma 1.3.8, Exercise 1.3.8.] *Let F be analytic on the open strip $S = \{z\in\mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and continuous on its closure such that*

$$\sup_{z\in\overline{S}} e^{-a|\mathrm{Im}(z)|}\log|F(z)| \le A < \infty \tag{2.43}$$

for some fixed A and $a < \pi$. Then, for all $0<\theta<1$, we have

$$\log|F(\theta)| \le \int_{\mathbb{R}} [\mu_0(t)\log|F(it)| + \mu_1(t)\log|F(1+it)|]\, dt. \tag{2.44}$$

We include the proof for the sake of convenience for readers.

Proof. For each $z\in\overline{S}$, define $H(z) := \log|F(z)|$ and $g(z) := \frac{e^{i\pi z}-i}{e^{i\pi z}+i}$. Let $\Delta(0,1) := \{z\in\mathbb{C} : |z|<1\}$. Observe that $g(z)\in\Delta(0,1)$ when $z\in S$ and that $g^{-1}(z) = \dfrac{1}{\pi i}\log\left(\dfrac{i(1+z)}{1-z}\right)$. Notice that $\mathrm{Im}(\frac{i(1+z)}{1-z}) > 0$ for every $z\in\Delta(0,1)$, so $z\in\Delta(0,1)\mapsto \log\left(\frac{i(1+z)}{1-z}\right)$ is a well-defined holomorphic function on $\Delta(0,1)$. Thus, g^{-1} maps $\Delta(0,1)$ conformally to S.

For each $z\in\Delta(0,1)$, define $G(z) := H(g^{-1}(z))$. Since $H(z)$ is subharmonic on S, we see that $G(z)$ is subharmonic on $\Delta(0,1)$. Then, for every $r\in(0,\rho)$ with $\rho<1$ and

$0 \le s \le 2\pi$, $G(re^{is}) \le \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2-r^2}{\rho^2-2\rho r\cos(t-s)+r^2} G(\rho e^{it})\ dt$. For every $\rho \in (r/2,1)$, we have

$$\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(t-s)+r^2}\mathrm{Re}(G(\rho e^{it})) \le \sup_{z\in\overline{S}} H(z)\frac{\rho^2-r^2}{\rho^2-2\rho r+r^2} = \sup_{z\in\overline{S}} H(z)\frac{\rho+r}{\rho-r}$$

and hence

$$\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(t-s)+r^2}\mathrm{Re}(G(\rho e^{it})) \le \sup_{z\in\overline{S}} H(z)\frac{1+r}{\frac{r+1}{2}-r} = \sup_{z\in\overline{S}} H(z)\frac{2+2r}{1-r}.$$

By the Fatou lemma and continuity of G, we get

$$\begin{aligned} G(re^{is}) &\le \limsup_{\rho\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(t-s)+r^2}G(\rho e^{it})\ dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-s)+r^2}G(e^{it})\ dt. \end{aligned}$$

For $\theta\in(0,1)$, we have $g(\theta) = \frac{e^{i\pi\theta}-i}{e^{i\pi\theta}+i} = -i\frac{\cos(\pi\theta)}{1+\sin\pi\theta}$, so, the solution of

$$re^{is} = g(\theta) \quad (r\in(0,1), s\in(0,2\pi))$$

is

$$(r,s) := \begin{cases} \left(\frac{\cos(\pi\theta)}{1+\sin(\pi\theta)}, \frac{3\pi}{2}\right) & \theta\in(0,1/2], \\ \left(-\frac{\cos(\pi\theta)}{1+\sin(\pi\theta)}, \frac{\pi}{2}\right) & \theta\in(1/2,1). \end{cases} \tag{2.45}$$

For (r,s) in (2.45), we have $H(\theta) = G(g(\theta)) \le \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-s)+r^2}H\left(g^{-1}(e^{it})\right)\ dt$ and $\frac{1-r^2}{1-2r\cos(t-s)+r^2} = \frac{\sin\pi\theta}{1+\sin t\cos(\pi\theta)}$, so

$$H(\theta) \le \frac{1}{2\pi}\int_0^{\pi}\frac{H(g^{-1}(e^{it}))\sin\pi\theta}{1+\sin t\cos(\pi\theta)}\ dt + \frac{1}{2\pi}\int_\pi^{2\pi}\frac{H(g^{-1}(e^{it}))\sin\pi\theta}{1+\sin t\cos(\pi\theta)}\ dt.$$

For $t\in[0,\pi]$, let $1+iy = g^{-1}(e^{it})$ with $y\in\mathbb{R}$. Then an arithmetic shows

$$e^{it} = g(1+iy) = -\tanh(\pi y) + i\mathrm{sech}(\pi y).$$

Consequently,

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{\pi}\frac{\sin\pi\theta}{1+\sin t\cos(\pi\theta)}H(g^{-1}(e^{it}))\ dt \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\pi\theta}{1+\mathrm{sech}(\pi y)\cos(\pi\theta)}H(1+iy)\ \pi\ \mathrm{sech}(\pi y)\ dy \\ &\quad = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(\pi\theta)}{\cosh(\pi y)+\cos(\pi\theta)}H(1+iy)\ dy. \end{aligned} \tag{2.46}$$

For $t \in [\pi, 2\pi]$, let $iy = g^{-1}(e^{it})$. Then $e^{it} = g(iy) = -\tanh(\pi y) - i\mathrm{sech}(\pi y)$. Therefore,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{\sin \pi\theta}{1+\sin t \cos(\pi\theta)} H(g^{-1}(e^{it}))\ dt \\
&\quad = \frac{1}{2\pi}\int_{\infty}^{-\infty} \frac{\sin \pi\theta}{1-\mathrm{sech}(\pi y)\cos(\pi\theta)} H(iy)\ (-\pi \mathrm{sech}(\pi y))\ dy \\
&\quad = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(\pi\theta)}{\cosh(\pi y)-\cos(\pi\theta)} H(iy)\ dy. \qquad (2.47)
\end{aligned}$$

By combining (2.46), (2.47), and $H(z) = \log|F(z)|$, we get the desired inequality. $\square$

Lemma 2.17. *Let $\theta \in (0,1)$. We define μ_0 and μ_1 by* (2.42)*. Then, for all functions F analytic on the open strip S and continuous on its closure satisfying* (2.43)*,*

$$|F(\theta)| \le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \mu_0(t)|F(it)|\, dt\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \mu_1(t)|F(1+it)|\, dt\right)^{\theta}. \qquad (2.48)$$

Proof. Let $j \in \{0,1\}$. Equipp $S_j \equiv \{(j,t) : t \in \mathbb{R}\}$ with a probability measure P_j given by $P_j(\{j\} \times E) \equiv \int_E \frac{\mu_j(t)}{\|\mu_j\|_{L^1(\mathbb{R})}}\ dt$ for any measurable set $E \subseteq \mathbb{R}$. We use (2.44) and the Jensen inequality to get

$$\begin{aligned}
&|F(\theta)| \\
&\le \left[\exp\left(\frac{1}{1-\theta}\int_{\mathbb{R}} \mu_0(t)\log|F(it)|\ dt\right)\right]^{1-\theta} \left[\exp\left(\frac{1}{\theta}\int_{\mathbb{R}} \mu_1(t)\log|F(1+it)|\ dt\right)\right]^{\theta} \\
&\le \left[\exp\left(\int_{S_0} \log|F(iT(\omega))|\ dP_0(\omega)\right)\right]^{1-\theta} \\
&\quad \times \left[\exp\left(\int_{S_1} \log|F(1+iT(\omega))|\ dP_1(\omega)\right)\right]^{\theta} \\
&\le \left(\int_{S_0} |F(iT(\omega))|\ dP_0(\omega)\right)^{1-\theta} \left(\int_{S_1} |F(1+iT(\omega))|\ dP_1(\omega)\right)^{\theta} \\
&= \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \mu_0(t)|F(it)|\ dt\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \mu_1(t)|F(1+it)|\ dt\right)^{\theta}.
\end{aligned}$$

Thus, (2.48) follows. $\square$

§ 2.3. Further remarks on the complex interpolation of Morrey spaces

Complex interpolation is a technique basically depending on the following three lines theorem.

Theorem 2.18. *Let $F : \bar{S} \to \mathbb{C}$ be a bounded continous function such that $F|S$ is holomorphic. Then $|F(\theta)| \le \|F(i\cdot)\|_{L^\infty(\mathbb{R})}^{1-\theta}\|F(1+i\cdot)\|_{L^\infty(\mathbb{R})}^{\theta}$ for all $0 < \theta < 1$.*

Based on Theorem 2.18, interpolation theory of Morrey spaces can be established. Despite a counterexample by Blasco, Ruiz and Vega [4, 31], interpolation theory of Morrey spaces progressed so much recently. As for the real interpolation results, Burenkov and Nursultanov obtained interpolation results in local Morrey spaces [5]. Nakai and Sobukawa generalized their results to B^u_w setting [27]. We made a significant progress in the complex interpolation theory of Morrey spaces. In [8, p. 35] Cobos, Peetre and Persson pointed out that $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subset \mathcal{M}^p_q$ as long as the parameters $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$ and $1 \le q \le p < \infty$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.49}$$

Note that (2.49) corresponds to (2.9). As is shown in [22, Theorem 3(ii)], when an interpolation functor F satisfies $F[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}] = \mathcal{M}^p_q$ under the condition (2.49), then

$$\frac{q_0}{p_0} = \frac{q_1}{p_1} \tag{2.50}$$

holds, using the counterexample by Ruiz and Vega [31]. Note again that (2.50) correpsonds to (2.8). Lemarié-Rieusset also proved that we can choose the second complex interpolation functor, introduced by Calderón in [6] in 1964. Meanwhile, as for the interpolation result under (2.49) and (2.50) by using the first complex interpolation functor by Calderón [6], Lu, Yang and Yuan obtained the following description: $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$ in [23, Theorem 1.2]. They also extended this result in the setting of a metric measure space. Their technique is again to calculate the Calderón product. The definition of Calderón product is given as follows: Let $\overline{X} = (X_0, X_1)$ be a compatible couple of Banach spaces and $\theta \in (0,1)$. The Calderón product ${X_0}^{1-\theta}{X_1}^\theta$ of X_0 and X_1 is defined by

$${X_0}^{1-\theta}{X_1}^\theta := \bigcup_{f_0 \in X_0, f_1 \in X_1} \{f : \mathbb{R}^n \to \mathbb{C} \, : \, |f(x)| \le |f_0(x)|^{1-\theta}|f_1(x)|^\theta\}.$$

For $f \in {X_0}^{1-\theta}{X_1}^\theta$, define the norm $\|f\|_{{X_0}^{1-\theta}{X_1}^\theta}$ by

$$\begin{aligned} &\|f\|_{{X_0}^{1-\theta}{X_1}^\theta} \\ &:= \inf\{\|f_0\|_{X_0}{}^{1-\theta}\|f_1\|_{X_1}{}^\theta \, : \, f_0 \in X_0, f_1 \in X_1, |f(x)| \le |f_0(x)|^{1-\theta}|f_1(x)|^\theta\}. \end{aligned}$$

We shall use the following density result:

Lemma 2.19. [6], [2, Theorem 4.2.2], [3] *Let X_0 and X_1 be a compatible couple of Banach spaces. Then $[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{{X_0}^{1-\theta}{X_1}^\theta} = \overline{X_0 \cap X_1}^{[X_0,X_1]^\theta}$. In particular, $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$.*

We refer to [44] for further extensions: Results are available to smoothness Morrey spaces described in [33, 43].

§ 3. Proofs

§ 3.1. Proof of Theorem 2.3

Let us prove (2.10) and admitting (2.11). By virtue of Lemma 2.19 and (2.11), we have

$$[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta = \overline{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)\cap\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}. \tag{3.1}$$

Let $f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$ be such that

$$\lim_{R\to\infty}\|f-f_R\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}=0. \tag{3.2}$$

Since $\|f_R\|_{\mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v)}\le R^{\frac{1}{q_j}}\|f\|^{\frac{q}{q_j}}_{\mathcal{M}^{\varphi}_{q}(w,v)}$ for every $j=0,1$, from (3.2) we deduce $f\in\overline{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)\cap\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}$. Therefore, by the identity (3.1), we conclude $f\in[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta$.

We remark that, for every $f\in\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)\cap\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$, we have

$$\begin{aligned}|f(x)\chi_{E_R}(x)|^q w(x) &\le |f(x)\chi_{E_{R,0}}(x)|^q w(x)+|f(x)\chi_{E_{R,1}}(x)|^q w(x)\\ &\le R^{-1}|f(x)|^{q_0}w_0(x)+R^{-1}|f(x)|^{q_1}w_1(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\|f-f_R\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)} &= \|f\chi_{E_R}\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}\\ &\le R^{-1}\|f\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)}+R^{-1}\|f\|_{\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}\to 0\end{aligned} \tag{3.3}$$

as $R\to\infty$. Hence,

$$\begin{aligned}&\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)\cap\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)\\ &\quad\subseteq\left\{f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\,:\,\lim_{R\to\infty}f_R=f\text{ in }\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\right\}.\end{aligned} \tag{3.4}$$

Thus, by combining (3.1) and (3.4), we obtain

$$\begin{aligned}&[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta\\ &\subset\left\{f\in\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\,:\,\lim_{R\to\infty}f_R=f\text{ in }\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)\right\}.\end{aligned}$$

Let us prove (2.11). Then, $f \in [\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta$ whenever $f \in \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$, according to Lemmas 2.10–2.13.

Conversely, suppose that $f \in [\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta$. Let $\varepsilon > 0$ be arbitrary. Then we can find $\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v))$ such that $f = G'(\theta)$ and that

$$\|G\|_{\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v))} \le (1+\varepsilon)\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta}.$$

In particular, the last inequality implies

$$\begin{aligned} &\varphi_j(x,r)\left(\frac{1}{v(B(x,r))}\int_{B(x,r)}|G(j+it,y)-G(j,y)|^{q_j}w_j(y)\,d\mu(y)\right)^{\frac{1}{q_j}} \\ &\le (1+\varepsilon)|t|\cdot\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta} \end{aligned} \tag{3.5}$$

for all $x \in X$, $r > 0$ and $t > 0$. Now, we fix $x \in X$, $r > 0$ and $j = 0,1$. For each $z \in \overline{S}$, define

$$H_t(z) := \frac{1}{v(B(x,r))^{\frac{1-z}{q_0}+\frac{z}{q_1}}}\left(\frac{G(z+it)-G(z)}{it}\right) w_0^{\frac{1-z}{q_0}} w_1^{\frac{z}{q_1}} \varphi_0(x,r)^{1-z}\varphi_1(x,r)^z. \tag{3.6}$$

Then, by combining (3.6) and (3.5), for each $t_0 \in \mathbb{R}$, we have

$$\|H_t(j+it_0,\cdot)\|_{L^{q_j}(B(x,r))} \le (1+\varepsilon)|t|\cdot\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta}.$$

Let a be a simple function such that $\|a\|_{L^{q'}(B(x,r))} = 1$ and

$$\|H_t(\theta,\cdot)\|_{L^q(B(x,r))} = \int_{B(x,r)} H_t(\theta,y)a(y)\;d\mu(y).$$

Define

$$F_t(z) := \int_{B(x,r)} H_t(z,y)\mathrm{sgn}(a(y))|a(y)|^{q'\left(\frac{1-z}{q_0'}+\frac{z}{q_1'}\right)}\;d\mu(y).$$

We use the Hölder inequality to obtain

$$|F_t(j+it_0)| \le (1+\varepsilon)|t|\cdot\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta}. \tag{3.7}$$

By virtue of the three line theorem and (3.7), we have

$$\|H_t(\theta+it_0,\cdot)\|_{L^q(B(x,r))} \le (1+\varepsilon)|t|\cdot\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta}$$

or equivalently,

$$\begin{aligned} &\varphi(x,r)\left(\frac{1}{v(B(x,r))}\int_{B(x,r)}|G(\theta+it,y)-G(\theta,y)|^{q}w(y)\,d\mu(y)\right)^{\frac{1}{q}} \\ &\le (1+\varepsilon)|t|\cdot\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta}. \end{aligned} \tag{3.8}$$

Write $K_t(\theta) := \frac{G(\theta+it)-G(\theta)}{t}$. We know that $K_t(\theta) \to f$ at least in the topology of $\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$. Since $\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) + \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v) \subseteq L^0(X,\mu)$, we can find a positive sequence $\{t_m\}_{m=1}^{\infty}$ decreasing to 0 such that

$$\lim_{m\to\infty} \chi_{B(x,r)}(y) K_{t_m}(\theta,y) = \chi_{B(x,r)}(y) f(y)$$

for almost every $y \in B(x,r)$. As a result, using the Fatou lemma and (3.8), we obtain

$$\varphi(x,r)\left(\frac{1}{v(B(x,r))}\int_{B(x,r)} |f(y)|^q w(y)\,d\mu(y)\right)^{\frac{1}{q}} \le (1+\varepsilon)\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta}.$$

Since $\varepsilon > 0$ is arbitrary, we see that $\|f\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)} \le \|f\|_{[\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]^\theta}$.

§ 3.2. Proof of Theorem 2.5

We prove (2.13). Let $f \in \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$. Define F, G, A, F_0, F_1, G_0 and G_1 by (2.29) and (2.30). The proof of $G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w}))$, amounts to establishing that $G_j(z) \in \mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v;\text{ rel }\tilde{w})$ and $G(j+it)-G(j) \in \mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v;\text{ rel }\tilde{w})$ for $j=0,1$, $z \in \overline{S}$, and $t \in \mathbb{R}$. Let $R > 1$. By Lemma 2.15 and

$$\chi_{\{\frac{1}{R}\le A\le R\}}|G_j(z)| \le \chi_{\{\frac{1}{R}\le A\le R\}}(1+|z|)|f|^{\frac{q}{q_j}} w^{\frac{1}{q_j}} {w_j}^{-\frac{1}{q_j}}, \tag{3.9}$$

we have $\chi_{\{\frac{1}{R}\le A\le R\}} G_j(z) \in \mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v;\text{ rel }\tilde{w})$. Moreover, since

$$\|G_j(z)(1-\chi_{\{\frac{1}{R}\le A\le R\}})\|_{\mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v)} \le 2(\log R)^{-1}\|f\|^{q/q_j}_{\mathcal{M}^{\varphi_j}_{q_j}} \to 0 \tag{3.10}$$

as $R \to \infty$, we conclude that $G_j(z) \in \mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v;\text{ rel }\tilde{w})$. Finally, since

$$\chi_{\{\frac{1}{R}\le A\le R\}}|G(it)-G(0)| = |t|\chi_{\{\frac{1}{R}\le A\le R\}}|f|^{\frac{q}{q_0}} w^{1/q_0} w_0^{-1/q_0} \tag{3.11}$$

and

$$\|(G(it)-G(0))\chi_{X\setminus\{\frac{1}{R}\le A\le R\}}\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)} \lesssim (\log R)^{-1}\|f\|^{q/q_0}_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)} \to 0 \tag{3.12}$$

as $R \to \infty$, we conclude $G(it)-G(0) \in \mathcal{M}^{\varphi_j}_{q_j}(X,\mu;w_j,v;\text{ rel }\tilde{w})$. We have an analogy of $G(1+it)-G(1)$ to (3.11) and (3.12). Hence, it follows that

$$G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w})).$$

Consequently, we obtain $f = G'(\theta) \in [\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w})]^\theta$.

Observe that $\overline{\mathcal{M}}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w}) \subseteq \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$ yields the first equality in (2.13). Meanwhile, $[\overline{\mathcal{M}}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w}),\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)]^\theta \subseteq \mathcal{M}^{\varphi}_{q}(X,\mu;w,v)$ as a consequence of (2.11).

We move on to the proof of (2.15). Let $f \in \mathcal{M}_q^{\varphi}(X,\mu;w,v)$ be such that $G^{(j)} \in \overset{*}{\mathcal{M}}{}_{q_j}^{\varphi_j}(X,\mu;w_j,v)$ for every $j=0,1$ and $R>1$. Define G, G_0, and G_1 as in (2.29) and (2.30). According to Lemmas 2.11–2.13, $G \in \mathcal{G}(\mathcal{M}_{q_0}^{\varphi_0}(X,\mu;w_0,v),\mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v))$. Moreover, arguing as in (3.9)–(3.12) and using $G^{(j)} \in \overset{*}{\mathcal{M}}{}_{q_j}^{\varphi_j}(X,\mu;w_j,v)$, we have

$$G(z) \in \widetilde{\mathcal{M}}_{q_0}^{\varphi_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}) + \widetilde{\mathcal{M}}_{q_1}^{\varphi_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})$$

for every $z \in \overline{S}$ and $G(j+it)-G(j) \in \widetilde{\mathcal{M}}_{q_j}^{\varphi_j}(X,\mu;w_j,v;\ \mathrm{rel}\ \tilde{w})$ for every $t \in \mathbb{R}$ and $j=0,1$. Thus, $G \in \mathcal{G}(\widetilde{\mathcal{M}}_{q_0}^{\varphi_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}),\widetilde{\mathcal{M}}_{q_1}^{\varphi_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w}))$. Consequently, $f=G'(\theta) \in [\widetilde{\mathcal{M}}_{q_0}^{\varphi_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w}),\widetilde{\mathcal{M}}_{q_1}^{\varphi_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})]^{\theta}$ as desired.

Next, let us show the last inclusion in (2.15). To this end, let

$$f \in [\overset{*}{\mathcal{M}}{}_{q_0}^{\varphi_0}(X,\mu;w_0,v),\mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v)]^{\theta}.$$

Then, by (2.11), $f \in \mathcal{M}_q^{\varphi}(X,\mu;w,v)$. Take $G \in \mathcal{G}(\overset{*}{\mathcal{M}}{}_{q_0}^{\varphi_0}(X,\mu;w_0,v),\mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v))$ such that $f=G'(\theta)$. Fix $x \in X$ and $r>0$. Let $B(\mathrm{o},R)$ be the ball as before. For every $z \in \overline{S}$ and $h \in \mathbb{R}\setminus\{0\}$, define

$$G_h(z) := \frac{\chi_{X\setminus B(\mathrm{o},R)}}{v(B(x,r))^{\frac{1-z}{q_0}+\frac{z}{q_1}}}\,(G(z+ih)-G(z))\,w_0^{\frac{1-z}{q_0}}\,w_1^{\frac{z}{q_1}}\,\varphi_0(x,r)^{1-z}\varphi_1(x,r)^{z}. \tag{3.13}$$

Let $H \in L^{\infty}(X)$ be such that $\mu\{H \neq 0\} < \infty$, $\|H\|_{L^{q'}(B(x,r))}=1$, and

$$\|G_h(\theta,\cdot)\chi_{B(x,r)}\|_{L^q(\mu)} = \int_{B(x,r)} G_h(\theta,y)H(y)\ d\mu(y). \tag{3.14}$$

For every $z \in \overline{S}$, set

$$f_{H,R}(z) := \int_{B(x,r)} G_h(z,y)\ \mathrm{sgn}(H(y))|H(y)|^{q'\left(\frac{1-z}{q_0'}+\frac{z}{q_1'}\right)}\,d\mu(y).$$

Then, by Lemma 2.17 and the Hölder inequality, we have

$$\begin{aligned}|f_{H,R}(\theta)| &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}}|f_{H,R}(it)|\mu_0(t)\,dt\right)^{1-\theta}\left(\frac{1}{\theta}\int_{\mathbb{R}}|f_{H,R}(1+it)|\mu_1(t)\,dt\right)^{\theta}\\ &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}}\|\chi_{X\setminus B(\mathrm{o},R)}(G(i(t+h))-G(it))\|_{\mathcal{M}_{q_0}^{\varphi_0}(X,\mu;w_0,v)}\mu_0(t)\,dt\right)^{1-\theta}\\ &\quad\times\left(\frac{1}{\theta}\int_{\mathbb{R}}\|G(1+i(t+h))-G(1+it)\|_{\mathcal{M}_{q_1}^{\varphi_1}(X,\mu;w_1,v)}\mu_1(t)\,dt\right)^{\theta}.\end{aligned}$$

As a consequence of the last inequality and (3.14), we have

$$\begin{aligned}
&\|\chi_{X\setminus B(o,R)}(G(\theta)-G(\theta+ih))\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}\\
&\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}}\|\chi_{X\setminus B(o,R)}(G(it)-G(it+ih))\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)}\mu_0(t)\,dt\right)^{1-\theta}\\
&\quad\times\left(\frac{1}{\theta}\int_{\mathbb{R}}\|G(1+it)-G(1+it+ih)\|_{\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}\mu_1(t)\,dt\right)^{\theta}.
\end{aligned}$$

Letting $R\to\infty$, we obtain $\|\chi_{X\setminus B(o,R)}(G(\theta)-G(\theta+ih))\|_{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}\downarrow 0$ as $R\to\infty$ thanks to the Lebesgue convergence theorem. Thus, $G(\theta+ih)-G(\theta)\in \overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v)$. Since $f=G'(\theta)$, that is,

$$\lim_{h\to 0^+}\left\|f-\frac{G(\theta+ih)-G(\theta)}{ih}\right\|_{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)+\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}=0,$$

we conclude that $f\in\overline{\overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v)}^{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v)+\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}$, which proves the last inclusion. Finally, remark that the remaining inclusions in (2.15) are consequences of trivial inclusion

$$\widetilde{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\ \mathrm{rel}\ \tilde{w})\subseteq\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$$

and $\widetilde{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\ \mathrm{rel}\ \tilde{w})\subseteq\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v)$.

The proof of (2.16) is similar to above and [15]. We omit the further detail.

§ 3.3. Proof of Theorem 2.6

We begin with the proof of (2.17). By virtue of Theorem 2.3 and also Lemmas 2.9 and 2.19, we obtain

$$\begin{aligned}
&[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta\\
&\quad\subseteq[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta\\
&\quad\subseteq\overline{\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v)\cap\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta}\\
&\quad\subseteq\overline{\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v)\cap\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}\\
&\quad\subseteq\overline{\overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v)}^{\mathcal{M}^{\varphi}_{q}(X,\mu;w,v)}=\overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v).
\end{aligned}$$

We combine this with (2.10) to obtain

$$\begin{aligned}
[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta&\subseteq[\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v),\mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta\\
&\subseteq\{f\in\overset{*}{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v):(2.7)\text{ holds}\}.
\end{aligned}$$

Conversely, let $f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q(X,\mu;w,v)$ be such that (2.7) holds. By the argument in the proof of (2.11), we have

$$f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q(X,\mu;w,v) \cap \overline{\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v) \cap \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{\mathcal{M}^{\varphi}_q(X,\mu;w,v)}.$$

Hence, $f \in [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta$ thanks to Lemmas 2.14 and 2.19.

We prove (2.18). From Theorem 2.3 and also Lemmas 2.9 and 2.19, we have

$$\begin{aligned}
&[\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}), \overline{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w})]_\theta\\
&\subset [\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta\\
&\subseteq \overline{\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}) \cap \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{[\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta}\\
&\subseteq \overline{\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}) \cap \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)}^{\mathcal{M}^{\varphi}_q(X,\mu;w,v)}\\
&\subseteq \overline{\overline{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w_0,v;\text{ rel }\tilde{w})}^{\mathcal{M}^{\varphi}_q(X,\mu;w,v)} = \overline{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w_0,v;\text{ rel }\tilde{w}).
\end{aligned}$$

Let $f \in \overline{\mathcal{M}}{}^{\varphi}_{q}(X,\mu;w,v;\text{ rel }\tilde{w})$ satisfy (2.7). Define $F_{0,R}$ and $F_{1,R}$ by replacing f with f_R in the definition of F_0 and F_1 in (2.31), respectively. Then $F_R = F_{0,R} + F_{1,R}$ goes to F in $\mathcal{F}(\mathcal{M}^{\varphi_0}_{q_0}(X,\mu;w_0,v), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v;))$, since f_R goes to f as $R \to \infty$ in $\mathcal{M}^{\varphi}_q(X,\mu;w_0,v)$. Since $F_R \in \mathcal{F}(\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}), \overline{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w}))$ for each $R > 0$ from the definition of E_R and Lemma 2.15, we have

$$F \in \mathcal{F}(\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}), \overline{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w})).$$

Thus

$$f \in [\overline{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{ rel }\tilde{w}), \overline{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{ rel }\tilde{w})]_\theta.$$

If we assume $w_0 = w_1 = w$ and reexamine the above proof, we can prove (2.19) similarly.

Next, we move on to the proof of (2.20). Let $f \in \widetilde{\mathcal{M}}{}^{\varphi}_q(X,\mu;w,v;\text{rel }\tilde{w})$ be such that (2.7) holds. By a similar argument as in the proof of the first equality in (2.18) and also Lemma 2.15 and $f \in \overset{*}{\mathcal{M}}{}^{\varphi}_q(X,\mu;w,v)$, we have

$$f \in [\widetilde{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{rel }\tilde{w}), \widetilde{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{rel }\tilde{w})]_\theta.$$

Observe that the second equality in (2.20) follows from $\widetilde{\mathcal{M}}{}^{\varphi_1}_{q_1}(X,\mu;w_1,v;\text{rel }\tilde{w}) \subseteq \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)$. Finally, by virtue of the second equality (2.17) and the second equality in (2.18), we have $f \in \widetilde{\mathcal{M}}{}^{\varphi}_q(X,\mu;w,v;\text{rel }\tilde{w})$ with (2.7) holds whenever $f \in [\widetilde{\mathcal{M}}{}^{\varphi_0}_{q_0}(X,\mu;w_0,v;\text{rel }\tilde{w}), \mathcal{M}^{\varphi_1}_{q_1}(X,\mu;w_1,v)]_\theta$. Finally, if we assume $w_0 = w_1 = w$ and reexamine the above proof, we can prove (2.21) similarly.

§ 4. Application to the boundedness of the Hardy-Littlewood maximal operatoron weighted Morrey spaces

In this appendix, we aim to summarize the recent progress of weight theory on Morrey spaces. To simplify the matters, we place ourselves in the setting of dyadic cubes on $\mathbb{R}^n$ equipped with the Lebesgue measure dx. Let $\mathcal{D}$ denote the set of all dyadic cubes of the form $2^{-j}m + [0, 2^{-j})^n$ for some $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Observe first that we can adopt the following equivalent norm when we consider the weighted Morrey space $\mathcal{M}^p_q(w,1) = \mathcal{M}^p_q(\mathbb{R}^n, dx; w, 1)$:

$$\|f\|_{\mathcal{M}^p_q(w,1)} = \sup_{Q\in\mathcal{D}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q w(x)\, dx\right)^{\frac{1}{q}}.$$

Accodingly the Hardy-Littlewood maximal operator M can be replaced by the dyadic maximal operator. From now on, we write $M_{\rm dyadic}$ for the dyadic maximal operator:

$$M_{\rm dyadic}f(x) := \sup_{Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q |f(y)|dy \cdot \chi_Q(x).$$

It is trivial that the Hardy-Littlewood maximal operator majorizes the dyadic maximal operator. Although the converse is not true, using the notion of the dyadic grid, we can justify that the dyadic maximal operator suffices when we consider Problem 1.3. In this setting, for $1 < p < \infty$ and a weight w, the constant $[w]_{A_p}$ must be modified: the constant $[w]_{A_p}$ adapted to this setting is:

$$[w]_{A_p} = \sup_{Q\in\mathcal{D}} \left(\frac{w(Q)}{|Q|}\right)\left(\frac{w^{-\frac{1}{p-1}}(Q)}{|Q|}\right)^{p-1}.$$

In addition, we define A_∞ to be the union of the class A_p where $1 < p < \infty$. The constant $[w]_{A_\infty}$ is $[w]_{A_\infty} = \sup_{Q\in\mathcal{D}} \frac{|Q|}{w(Q)} \exp\left(\frac{1}{|Q|}\int_Q \log w(x)\, dx\right)$.

One of the crucial properties of the class A_∞ is that

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}\, dx\right)^{\frac{1}{1+\varepsilon}} \le \frac{2}{|Q|}\int_Q w(x)\, dx \quad \left(\varepsilon := \frac{1}{2^{n+3}[w]_{A_\infty}}\right). \tag{4.1}$$

A direct consequence of (4.1) is that for $\alpha \ll 1$ there exists $\beta = \beta(\alpha, [w]_{A_\infty}) \in (0,1)$ such that

$$|E| \le \alpha|Q| \Longrightarrow w(E) \le \beta w(Q) \tag{4.2}$$

for all measurable sets E and cubes Q with $E \subset Q$. In [28], we introduced the weight class $\mathcal{B}_{p,q}$ in the context of the boundedness of the Hardy-Littlewood maximal operator on weighted Morrey spaces. We define $\Phi_{p,q,w}(Q) := |Q|^{\frac{1}{p}}\left(\frac{w(Q)}{|Q|}\right)^{\frac{1}{q}}$ for $Q \in \mathcal{Q}$.

Definition 4.1 ([28]). Let $1 \le q \le p < \infty$ and w be a weight. One says that a weight w is in the class $\mathcal{B}_{p,q}$ if there exists $C_{p,q} > 0$ such that for any $Q_0 \in \mathcal{Q}$,

$$\sup_{Q \in \mathcal{Q}: Q \subset Q_0} \Phi_{p,q,w}(Q) \le C_{p,q} \Phi_{p,q,w}(Q_0), \tag{4.3}$$

or equivalently, $\|\chi_{Q_0}\|_{\mathcal{M}^p_q(dx,w)} \sim \Phi_{p,q,w}(Q_0)$ hold.

We may state the following partial answer to Problem 1.3 based on the above setup.

Theorem 4.2 ([28]).
Let $1 < q \le p < \infty$ and $w \in A_q \cap \mathcal{B}_{p,q}$. Then M_{dyadic} is bounded from $\mathcal{M}^p_q(w)$ to $\mathcal{M}^p_q(w)$.

Although the above gives a sufficient condition, one can notice that the condition is too strong. Indeed, Tanaka [35] gave the characterization of the boundedness of M on $\mathcal{M}^p_q(w)$ with power weight $w(x) = |x|^\alpha$ as follows:

Theorem 4.3 ([35]).
Let $1 < q < p < \infty$ and $w_\alpha(x) = |x|^\alpha$ with $\alpha > -n$. Then M is bounded on $\mathcal{M}^p_q(w_\alpha)$ if and only if

$$-n\frac{q}{p} = -n + n\left(1 - \frac{q}{p}\right) \le \alpha < n(q-1) + n\left(1 - \frac{q}{p}\right) = nq\left(1 - \frac{1}{p}\right).$$

Theorem 4.3 tells us that the condition A_q in Theorem 4.2 is too strong. Meanwhile, for the Hilbert transform defined by

$$Hf(x) := \mathrm{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy \quad (x \in \mathbb{R}),$$

Samko [30] showed that H is bounded on $\mathcal{M}^p_q(w_\alpha)$ if and only if $-\frac{q}{p} < \alpha < q\left(1 - \frac{1}{p}\right)$.

From these points of view, it is natural to ask ourselves what is the necessary and sufficient condition imposed on weight w for which M is bounded on $\mathcal{M}^p_q(w)$. To tackle this problem, we here give some necessary condition.

Recall that the weight w satisfies the doubling condition if there exists a constant $C > 0$ such that for any $Q \in \mathcal{Q}$, $w(2Q) \le Cw(Q)$ holds.

Theorem 4.4. *Let $1 < q \le p < \infty$ and w be a weight. Assume that M_{dyadic} is bounded on $\mathcal{M}^p_q(dx, w)$. Then we have the following:*

1. $w \in \mathcal{B}_{p,q} \cap A_{q+1}$.
2. $w^{-\frac{1}{q-1}} \in \mathcal{B}_{p,q}$ *implies* $w \in A_q$.

3. $w^{-\frac{1}{q-1}} \in A_\infty$ *is equivalent to* $w \in A_q$.

Proof.

1. First, for the proof of the necessity of $w \in \mathcal{B}_{p,q}$, we refer to [29].

   Next, let us show $w \in A_{q+1}$. To this end, we write $\sigma^* := w^{-\frac{1}{q}}$. We fix $Q_0 \in \mathcal{D}$ and calculate the norm of $\chi_{Q_0} \cdot \sigma^*$: $\|\chi_{Q_0} \cdot \sigma^*\|_{\mathcal{M}^p_q(dx,w)} = \|\chi_{Q_0}\|_{\mathcal{M}^p_q(dx,dx)} = |Q_0|^{\frac{1}{p}}$. Since, we have the pointwise estimate: $\chi_{Q_0}(x)\frac{\sigma^*(Q_0)}{|Q_0|} \le M_{\rm dyadic}[\chi_{Q_0} \cdot \sigma^*](x)$ for $x \in \mathbb{R}^n$, the boundedness of $M_{\rm dyadic}$ implies that

$$\frac{\sigma^*(Q_0)}{|Q_0|}\|\chi_{Q_0}\|_{\mathcal{M}^p_q(dx,w)} \le \|M_{\rm dyadic}\|_{\mathcal{M}^p_q(dx,w)\to\mathcal{M}^p_q(dx,w)}\|\chi_{Q_0}\sigma^*\|_{\mathcal{M}^p_q(dx,w)} \sim |Q_0|^{\frac{1}{p}}.$$

   If we notice that

$$\|\chi_{Q_0}\|_{\mathcal{M}^p_q(dx,w)} \ge \Phi_{p,q,w}(Q_0), \tag{4.4}$$

   then $\left(\frac{w(Q_0)}{|Q_0|}\right)^{\frac{1}{q}} \frac{\sigma^*(Q_0)}{|Q_0|} \le \|M_{\rm dyadic}\|_{\mathcal{M}^p_q(dx,w)\to\mathcal{M}^p_q(dx,w)}$, which implies $w \in A_{q+1}$ with $[w]_{A_{q+1}} \le \|M_{\rm dyadic}\|^{\frac{1}{q}}_{\mathcal{M}^p_q(dx,w)\to\mathcal{M}^p_q(dx,w)}$.

2. Let us show $w \in A_q$. To this end, we fix any $Q \in \mathcal{D}$ and write $\sigma := w^{-\frac{1}{q-1}}$. Since $\sigma \in \mathcal{B}_{p,q}$ is a dual weight of w: $\sigma^q w = \sigma$, we notice that

$$\|\chi_Q \cdot \sigma\|_{\mathcal{M}^p_q(dx,w)} = \|\chi_Q\|_{\mathcal{M}^p_q(dx,\sigma)} \sim |Q|^{\frac{1}{p}} \left(\frac{\sigma(Q)}{|Q|}\right)^{\frac{1}{q}}.$$

   On the other hand, since $\frac{\sigma(Q)}{|Q|}\chi_Q(x) \le M_{\rm dyadic}[\chi_Q \cdot \sigma](x)$ for all $x \in \mathbb{R}^n$, the boundedness of $M_{\rm dyadic}$ on $\mathcal{M}^p_q(dx,w)$ yields that

$$\frac{\sigma(Q)}{|Q|}\|\chi_Q\|_{\mathcal{M}^p_q(dx,w)} \le \|M_{\rm dyadic}[\chi_Q \cdot \sigma]\|_{\mathcal{M}^p_q(dx,w)} \lesssim \|\chi_Q \cdot \sigma\|_{\mathcal{M}^p_q(dx,w)} \sim |Q|^{\frac{1}{p}} \left(\frac{\sigma(Q)}{|Q|}\right)^{\frac{1}{q}}.$$

   Moreover, since we know that

$$\|\chi_Q\|_{\mathcal{M}^p_q(dx,w)} \ge \frac{w(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}-\frac{1}{q}}}.$$

   by dividing the both terms by $|Q|^{\frac{1}{p}}$, it follows that $\frac{\sigma(Q)}{|Q|}\left(\frac{w(Q)}{|Q|}\right)^{\frac{1}{q}} \le C_0 \left(\frac{\sigma(Q)}{|Q|}\right)^{\frac{1}{q}}$, or equivalently,

$$\frac{w(Q)}{|Q|}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{q-1}}\,dx\right)^{q-1} \le C_0^q,$$

   which implies $w \in A_q$.

3. Let us show $w \in A_q$. Fix any $Q_0 \in \mathcal{D}$ and set $\gamma_0 := \frac{1}{|Q_0|}\int_{Q_0} \sigma(y)dy$ and take a large constant $a_{[\sigma]_{A_\infty}} > 2^n$ so that $2^n/a_{[\sigma]_{A_\infty}} \le \lambda'_\sigma \ll 1$. Then we define $\mathcal{D}_0 := \{Q_0\}$ and

$$\mathcal{D}_k := \left\{ Q \in \mathcal{D}(Q_0) : \frac{\sigma(Q)}{|Q|} > a_{[\sigma]_{A_\infty}}^k \gamma_0 \right\} \quad (k \in \mathbb{N}).$$

We denote the maximal subset of $\mathcal{D}_k$ by $\mathcal{D}_k^* := \{Q_j^k\}_{j \in J_k}$ again. Notice that

$$|Q_j^k \cap \Omega_{k+1}| \le \frac{2^n}{a_{[\sigma]_{A_\infty}}} |Q_j^k|.$$

From this, (4.2) and $2^n/a_{[\sigma]_{A_\infty}} \le \lambda'_\sigma \ll 1$, we see that $\{Q_j^k\}_{k \in \mathbb{N}_0, j \in J_k}$ is a σ-sparse family. In particular, it follows that $\sigma(Q_0 \setminus \Omega_1) \ge C_{[\sigma]_{A_\infty}} \sigma(Q_0)$, where $\Omega_1 := \bigcup_{j \in J_1} Q_j^1 = \bigcup_{R \in \mathcal{D}_1} R$. This implies that

$$\chi_{Q_0}(x) \frac{\sigma(Q_0)}{|Q_0|} \lesssim_{[\sigma]_{A_\infty}} \chi_{Q_0}(x) \frac{\sigma(Q_0 \setminus \Omega_1)}{|Q_0|} \le M_{\text{dyadic}}[\chi_{Q_0 \setminus \Omega_1} \cdot \sigma](x).$$

By taking the weighted Morrey norm of both sides and using the boundedness of M_{dyadic} and $\sigma^q \cdot w = \sigma$, we learn

$$\frac{\sigma(Q_0)}{|Q_0|} \|\chi_{Q_0}\|_{\mathcal{M}^p_q(dx,w)} \lesssim_{[\sigma]_{A_\infty}} \|\chi_{Q_0 \setminus \Omega_1} \cdot \sigma\|_{\mathcal{M}^p_q(dx,w)} = \|\chi_{Q_0 \setminus \Omega_1}\|_{\mathcal{M}^p_q(dx,\sigma)}. \tag{4.5}$$

By recalling that

$$\Omega_1 = \bigcup_{R \in \mathcal{D}_1} R, \quad \mathcal{D}_1 := \left\{ R \in \mathcal{D}(Q_0) : \frac{\sigma(R)}{|R|} > a_{[\sigma]_{A_\infty}} \frac{\sigma(Q_0)}{|Q_0|} \right\},$$

we see that

$$\|\chi_{Q_0 \setminus \Omega_1}\|_{\mathcal{M}^p_q(dx,\sigma)} \le a_{[\sigma]_{A_\infty}}^{\frac{1}{q}} |Q_0|^{\frac{1}{p}} \left(\frac{\sigma(Q_0)}{|Q_0|} \right)^{\frac{1}{q}}. \tag{4.6}$$

Meanwhile, we know that

$$\|\chi_{Q_0}\|_{\mathcal{M}^p_q(dx,w)} \ge |Q_0|^{\frac{1}{p}} \left(\frac{w(Q_0)}{|Q_0|} \right)^{\frac{1}{q}}. \tag{4.7}$$

As a result, by inserting (4.6) and (4.7) into (4.5), we obtain that

$$\frac{\sigma(Q_0)}{|Q_0|} \left(\frac{w(Q_0)}{|Q_0|} \right)^{\frac{1}{q}} \lesssim_{[\sigma]_{A_\infty}} \left(\frac{\sigma(Q_0)}{|Q_0|} \right)^{\frac{1}{q}},$$

which implies $w \in A_q$.

□

Recall that $w \in A_q$ is equivalent to $w, \sigma = w^{-\frac{1}{q-1}} \in A_\infty$ under the boundedness of the maximal operator. From this point of view, in the case of Lebesgue spaces, the condition $\sigma \in A_\infty$ was natural one. However, Theorem 4.4 shows that even the condition $\sigma \in A_\infty$ is too strong.

Finally we end this paper with a result related to the complex interpolation. We go back to the classical Hardy-Littlewood maximal operator from the dyadic maximal operator $M_{\rm dyadic}$.

Theorem 4.5. *Maintain the same conditions as Theorem 2.3. Assume in addition that $q_0, q_1 > 1$ and that (X, d, μ) is the Euclidean space $(\mathbb{R}^n, |\cdot|, dx)$. If the Hardy-Littlewood maximal operator M is bounded on $\mathcal{M}^{\varphi_i}_{q_i}(w_i, v)$ for $i = 0, 1$, then M is bounded on $\mathcal{M}^{\varphi}_q(w, v)$.*

Proof. We linearlize M: Let N be a bounded measurable functions and $\{E_j\}_{j=1}^\infty$ be a partition of $\mathbb{R}^n$. We define $M^*f = \sum_{j=1}^\infty \left(\frac{1}{B(x,N(\cdot))}\int_{B(x,N(\cdot))} f(y)\,dy\right)\chi_{E_j}$ for $f \in L^1_{\rm loc}$. We have only to prove that M^* is bounded on $\mathcal{M}^{\varphi}_q(w, v)$. We know that M^* is bounded on $\mathcal{M}^{\varphi_i}_{q_i}(w_i, v)$ by assumption. Thus, by virute of (2.11) we conclude that M^* is bounded on $\mathcal{M}^{\varphi}_q(w, v)$. □

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